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Functional Relations for the Order Parameters of the Chiral Potts Model

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Following the method of Jimbo, Miwa, and others, we obtain functional relations for the order parameters of the chiral Potts model. We have not yet solved these relations. Here we discuss their properties and show how one should beware of spurious solutions.

KEY WORDS: Statistical mechanics; lattice models; chiral Potts model; spontaneous magnetization; functional relations.

1. INTRODUCTION

An outstanding problem in statistical mechanical lattice models remains the calculation of the order parameters of the two-dimensional solvable chiral N-state Potts model. This model was originally formulated as a one-dimensional quantum hamiltonian[1-3] and later as a two-dimensional classical lattice model in statistical mechanics.[4-6] It is "solvable" in the sense that it satisfies the star-triangle relations, and indeed the free energy has been calculated.[7-9]

For previously solved models, the order parameters can also be readily calculated by using the corner transfer matrix approach.[10] Surprisingly, this method completely fails for the chiral Potts model, due to the lack of the difference property.[11] This situation is particularly tantalising because there is an intriguingly simple conjecture for the order parameter. If a is a spin deep inside the lattice and $\omega = \exp(2\pi i/N)$, then for $r = 0,..., N$,

$$\mathscr{M}_r = \langle \omega^{ra} \rangle = (1 - k'^2)^{r(N-r)/2N^2} \tag{1}$$

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(Eq. 3.13 of ref. 2, Eq. 1.20 of ref. 12, Eq. 15 of ref. 13, β and λ therein being the k' of this paper; the system is to be ferromagnetically ordered, which implies $0 < k' < 1$). This k' is a temperature-like variable: it varies from zero at zero temperature to 1 at criticality. For $N = 3$ the conjecture (1) has been verified by Howes *et al.*[2] to order k'^{13}, and to order k'^{15} by the author.[11] For general N it has been verified to order k'^5.[12]

Jimbo, Miwa, and Nakayashiki[14] have invented an alternative (but related) method, which we may call the "broken rapidity line" method. This has been developed by others,[15–18] including Davies and Peschel.[19] In this one generalizes the local one-spin correlations and derives functional relations for the generalized function.

Here we obtain these relations for a general solvable edge-interaction model. We specialize these firstly to models with Z_N-symmetry, and then to the chiral Potts model. The resulting equations have been reported in ref. 20.

Such functional relations are sometimes referred to as "difference equations." This is rather misleading as difference equations usually apply to functions whose domain is a set of integers,[21] whereas these equations have the complex plane as their domain. They resemble the periodicity relation $f(z+1) = f(z)$, rather than the difference equation $f(n+1) = f(n)$. Such an equation does not define $f(z)$: it merely imposes the condition that $f(z)$ be periodic of period 1. One needs further information in some basic domain: for instance, if one is told that $f(z)$ is analytic in the vertical strip $0 \leqslant \operatorname{Re} z \leqslant 1$ and its derivative is Fourier analyzable in the vertical direction inside this strip, then it follows that $f(z)$ is a constant. Another such functional relation is the inversion relation (or unitarity condition) for the free energy.[10, 22] As with this relation, we need appropriate analyticity information in some basic domain for the relations to define the function. At present we do not have such information for the chiral Potts model.

Another problem is a technical one. For previously solved models with the rapidity difference property, one can solve the relations by Fourier transforms or Laurent expansions: such a technique has not as yet been developed for the chiral Potts model.

As an illustration of the first difficulty, we present a simple algebraic function which satisfies all the relations but yields a result for $\mathscr{M}_r$ which differs by a power $N/2$ from the conjecture (1). This difference is manifest at order k'^2, to which order (1) is certainly correct. The algebraic solution is therefore wrong. (There are other examples in the literature of finding wrong solutions of functional relations: see ref. 23.)

It does seem likely that the generalized correlation function is a meromorphic function on some extended Riemann surface, with poles and zeros only at certain known locations. We discuss a possible structure of these poles and zeros. In a subsequent paper we intend to discuss the hyperelliptic function parametrization of the relations for the $N=3$ case,[24, 25] and present some short series expansion results that we hope will assist the search for the correct solution.

2. FORMULATION

A quite general way of formulating a solvable edge-interaction model is to consider a set of directed lines forming a graph $\mathscr{L}$ of the type shown in Fig. 1.

The lines (in this case six) head generally from the bottom of the graph to the top, intersecting one another on the way. They can go locally downwards, but there can be no closed directed paths in $\mathscr{L}$. (This means that one can always distort $\mathscr{L}$, without changing its topology, so that lines always head upwards.) In particular, two lines can cross as in Fig. 2a, but not as in Fig. 2b.

Now shade alternate faces of $\mathscr{L}$, as in Fig. 3. Form another graph $\mathscr{G}$ by placing a site in each unshaded face, with edges connecting sites belonging to faces that touch at a corner. These sites and edges of $\mathscr{G}$ are represented in Fig. 3 by circles and heavy solid lines, respectively. For each intersection of lines in $\mathscr{L}$, there is an edge of $\mathscr{G}$ passing through it, and conversely. $\mathscr{L}$ is the "medial" graph of $\mathscr{G}$ (pp. 47, 124 of ref. 26).

Now we define a statistical mechanical spin model on $\mathscr{G}$. With each line of $\mathscr{L}$ associate a "rapidity" p. On each site i of $\mathscr{G}$ place a spin σ_i, taking some set of values: here we shall take $\sigma_i = 0, \ldots, N-1$, where N is the number of states available to each spin.

Each edge of G is either of the first type in Fig. 4, or the second. Let a, b be the spins at the end sites and p, q the rapidities of the associated

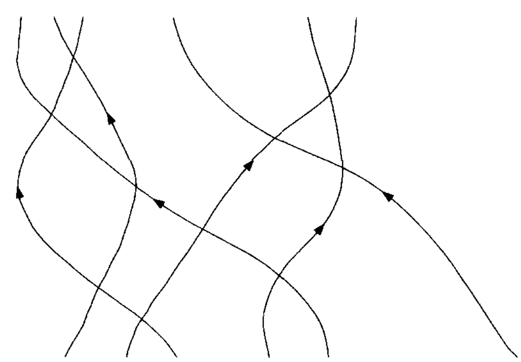

Fig. 1. A set of directed lines going from the bottom to the top of the figure.

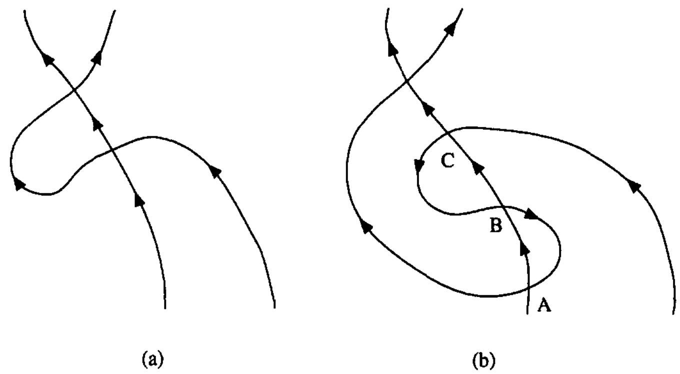

Fig. 2. (a) An allowed line configuration; (b) a non-allowed configuration: there is a closed directed path from A to B to A, and another from B to C to B.

lines of $\mathscr{L}$, arranged as in Fig. 4. Then if the edge is of the first type, the spins a, b interact with Boltzmann weight function $W_{pq}(a, b)$. If the edge is of the second type, they interact with weight $\bar{W}_{pq}(a, b)$. In general there may also be a self-interaction rapidity-independent weight $S(a)$ for each spin a (as in the Kashiwira-Miwa model[27]). The partition function is

$$Z = \sum \prod_{i} S(\sigma_i) \prod_{(i,j)} W_{pq}(\sigma_i, \sigma_j) \prod_{(k,l)} \bar{W}_{pq}(\sigma_k, \sigma_l) \tag{2}$$

where the first product is over all sites i, the second is over all edges (i, j) of the first type, the third over all edges (k, l) of the second type, and the sum is over all values of all the spins.

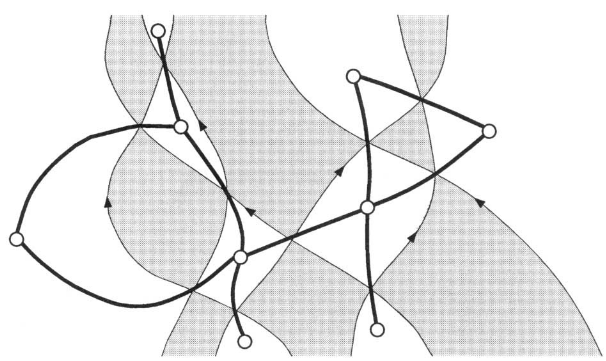

Fig. 3. The graph $\mathscr{G}$ formed by shading alternate faced of Fig. 1 and putting sites on the unshaded faces.

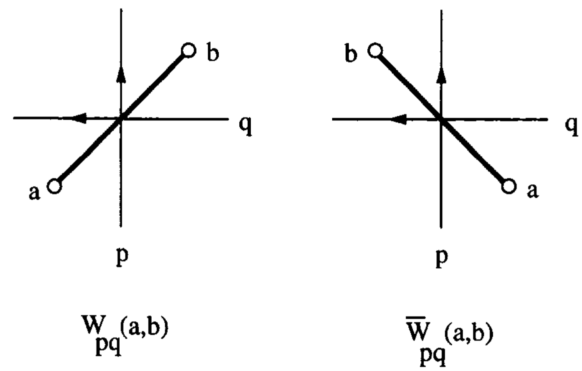

Fig. 4. The two types of edge on $\mathscr{G}$.

Averages (or expectation values) are calculated in the usual way. In particular, the expectation value of a function f of the spin σ_0 is

$$\langle f(\sigma_0)\rangle = Z^{-1} \sum f(\sigma_0) \prod_i S(\sigma_i) \prod W_{pq}(\sigma_i, \sigma_j) \prod \overline{W}_{pq}(\sigma_k, \sigma_l) \tag{3}$$

In this paper we consider a ferromagnetic model, where like adjacent spins are energetically favoured, and in which one of the possible ground states has all spins zero. To favour this state we set all boundary spins to be zero. (In general the boundary spins should be set to their values in the chosen ground state.)

3.1. Star-Triangle Relations

We require that the functions W, $\overline{W}$ satisfy the two star-triangle relations[6]

$$\sum_{d=0}^{N-1} S(d)\, \overline{W}_{qr}(b, d)\, W_{pr}(a, d)\, \overline{W}_{pq}(d, c) = R_{pqr}\, W_{pq}(a, b)\, \overline{W}_{pr}(b, c)\, W_{qr}(a, c)$$

$$\sum_{d=0}^{N-1} S(d)\, \overline{W}_{qr}(d, b)\, W_{pr}(d, a)\, \overline{W}_{pq}(c, d) = R_{pqr}\, W_{pq}(b, a)\, \overline{W}_{pr}(c, b)\, W_{qr}(c, a) \tag{4}$$

for all values of the three spins a, b, c and all rapidities p, q, r. The sides of the first relation are the partition functions of the two graphs in Fig. 5, the "external" spins a, b, c being held fixed. For the second relation, reverse all arrows.

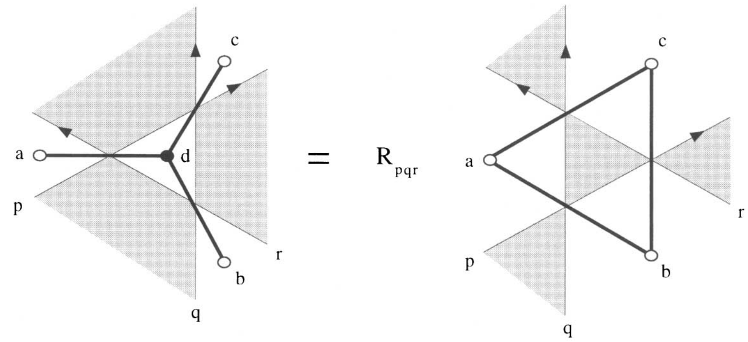

Fig. 5. The first of the star–triangle relations (4).

We also require that the the functions satisfy the inverse relations

$$\sum_{c=0}^{N-1} S(c)\, \bar{W}_{pq}(a, c)\, \bar{W}_{qp}(c, b) = \lambda_{pq} \delta(a, b) \qquad (5)$$
$$W_{pq}(a, b)\, W_{qp}(a, b) = 1$$

These relations are depicted graphically in Figs. 6 and 7.

The spin-independent factors R_{pqr}, λ_{pq} can be obtained by regarding the equations as the elements (b, c) or (a, b) of a matrix equation, any other external spin being held fixed, writing the expressions in terms of matrix products and taking determinants.[28] This gives

$$R_{pqr} = \hat{S} f_{pq} f_{qr} / f_{pr}, \qquad \lambda_{pq} = \hat{S} f_{pq} f_{qp} \qquad (6)$$

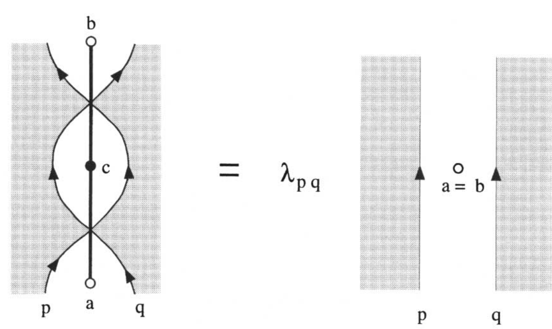

Fig. 6. The first inversion relation (5).

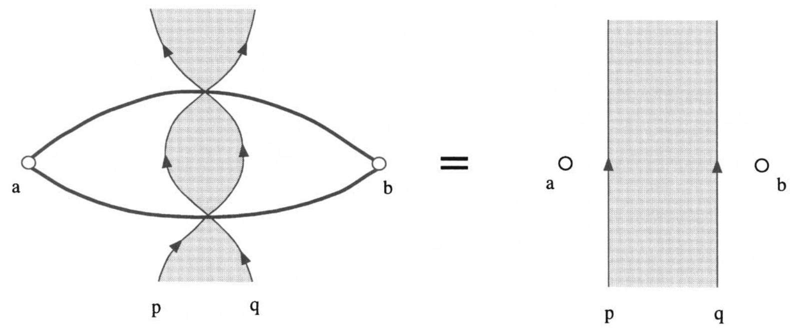

Fig. 7. The second inversion relation (5).

where

$$\hat{S}=[S(0)\cdots S(N-1)]^{1/N}, \qquad f_{pq}=(\det{}_N \bar{W}_{pq})^{1/N} \prod_{a=0}^{N-1} \prod_{b=0}^{N-1} W_{pq}(a,b)^{-1/N^2} \tag{7}$$

In fact it imposes restrictions on the products of $W_{pq}(a, b)$ over b, for given a (and over a, for given b). These restrictions are satisfied automatically for the Z_N-invariant models we shall be considering, so we do not dwell on this point.

The vital point about these relations (4) and (5) is that they ensure that the partition function Z is unchanged (apart from simple $\hat{S}$ and f_{pq}) factors by continuously moving the the lines of $\mathscr{L}$ (keeping their boundary positions fixed), *so long as* $\mathscr{L}$ *remains directed.*

This last qualification is important: no closed directed paths are allowed to appear, as in Fig. 2b. Note that the triangle in Fig. 5 does *not* contain such a path: there is no star-triangle relation where the arrows follow one another round the central rapidity triangle. Nor is there any inversion relation where the arrows follow one another round the central two-sided face, as they do in parts of Fig. 2b.

The expectation value $\langle f(\sigma_0)\rangle$ is also unchanged by such moves—in fact strictly unchanged because the $\hat{S}$ and f_{pq} factors cancel out of Eq. 3—provided that no line moves across the spin σ_0. If σ_0 is deep inside a very large graph $\mathscr{L}$, then any given line can still be moved an arbitrarily long way from σ_0. If all the Boltzmann weights are positive real, then by physical arguments we expect $\langle f(\sigma_0)\rangle$ to be independent of effects far removed from site 0, and hence in this large-lattice limit

$$\langle f(\sigma_0)\rangle = \text{independent of all rapidities} \tag{8}$$

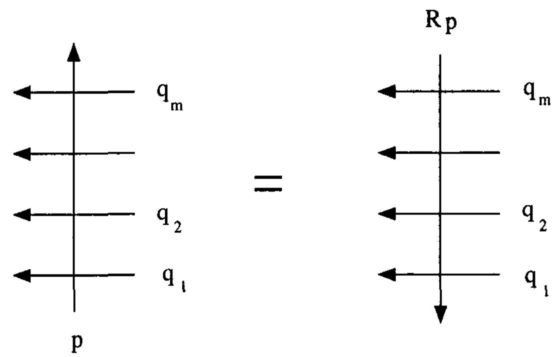

Fig. 8. An allowed arrow reversal: note that the lines $q_1, \ldots, q_m$ must *all* point to the left.

Similarly, $\langle f(\sigma_0) \rangle$ should be independent of the arrangement of the lines in $\mathscr{L}$, so it should be "universal": the same for any allowed graph: for instance $\mathscr{G}$ may be the square, triangular or honeycomb lattice. (These arguments are given in less generality in ref. 29.)

While this makes it all the more interesting to calculate $\langle f(\sigma_0) \rangle$, it presents a difficulty: it is hard to see how one can write down a functional equation that at least partially defines $\langle f(\sigma_0) \rangle$ (for the free energy such a relation follows easily from the inversion relations in Eq. 5). A solution to this problem has been given by ref. 14 and will be discussed in the next section. First we need one more property of W_{pq} and $\bar{W}_{pq}$: there exists an operation R which takes a rapidity p to another rapidity Rp such that

$$\bar{W}_{pq}(a, b) = W_{q, Rp}(a, b), \qquad W_{pq}(a, b) = \bar{W}_{q, Rp}(b, a) \tag{9}$$

It follows that the weights in Fig. 4 are unchanged by reversing the vertical arrow and changing p to Rp (keeping the spins a, b fixed). The two directed line configurations in Fig. 8 are therefore equivalent (for both of the possible alternate shadings of the faces).

3. GENERALIZED LOCAL CORRELATION FUNCTION

Now we focus on the square lattice. In Fig. 9 we have drawn a square lattice $\mathscr{L}$ of rapidity lines. It is directed: all lines are directed generally from the SE up to the NW (to align with Fig. 1, rotate through 45°). The associated graph $\mathscr{G}$ is to be drawn on the unshaded faces of $\mathscr{L}$: we have indicated only one site, with spin a. If this spin is held fixed, the partition

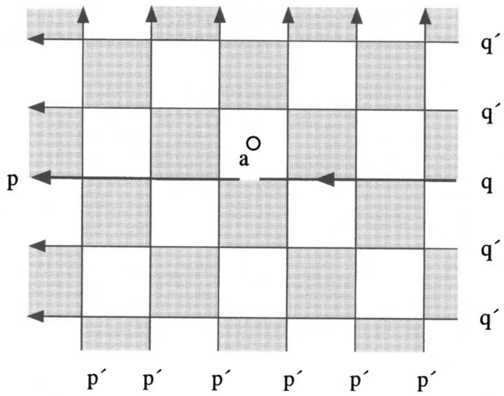

Fig. 9. The square lattice with a broken horizontal rapidity line.

function is a function $Z(a)$ of a. If it is allowed to take all values, the probability that the spin has value a is

$$F(a) = Z(a)/[Z(0) + \cdots Z(N-1)] \tag{10}$$

Jimbo, Miwa, and Nakayashiki proposed an ingenious trick.[14-18] We use this here, our approach being related to that of Davies and Peschel.[19] Consider the horizontal line immediately below the spin a. We break this in the middle, immediately adjacent to a, as indicated in the figure. Let us call the two resulting line segments the "special" lines. Assign different rapidities p, q to them, and rapidities p', q' to the other "background" lines, as indicated in Fig. 9. Then any of the background lines can be shifted (by a sequence of allowed star–triangle and inversion crossing moves (4) and (5)) to the boundary, so in the limit when $\mathscr{L}$ is large we expect $F(a)$ to be independent of p' and q'.

On the other hand, the break in the special line prevents us from moving it away from a (in fact the break must remain adjacent to a), so $F(a)$ can and does depend on p and q:

$$F(a) = F_{pq}(a) \tag{11}$$

Of course, if $p = q$, then there is no need for the break in the special line, so it too can then be removed to the boundary, giving

$$F_{pp}(a) = \text{independent of } p \tag{12}$$

This function $F_{pp}(a)$ is the usual probability of a central spin being in state a, so

$$\langle f(\sigma_0)\rangle = \langle f(a)\rangle = \sum_{a=0}^{N-1} f(a)\, F_{pp}(a) \tag{13}$$

3.1. Rotation Symmetries

We return to considering the general situation, when the rapidities p, q of the special line segments are different and their join has to remain adjacent to a. We establish three rotation symmetries, which are functional relations satisfied by $F_{pq}(a)$.

We are permitted to *rotate* them around a, so long as we take care to maintain $\mathscr{L}$ as a directed graph. In particular no two anti-parallel lines must be allowed to cross, as in Fig. 2b. More precisely, we can deform the special line segment q as in Fig. 10, still maintaining $\mathscr{G}$ as a directed graph. By pushing the upper and right-hand parts of this deformed curve to the boundary, the effect is to rotate q through 90° in the widdershins direction, as in Fig. 11. (From now on we only show the special line segments in our figures, omitting the background lines, except in Fig. 14.)

The line segment q then points downwards, so is anti-parallel to all the background vertical lines. This means we cannot push it any further to the left.

To get round this difficulty we use the equivalence of Fig. 8, and reverse the direction of the arrow, while also replacing q by $R^{-1}q$, as in

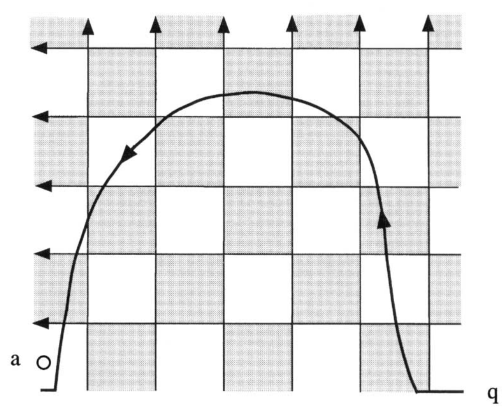

Fig. 10. An allowed deformation of the line segment q into the upper-right quadrant. One can keep pushing it up and to the right until it consists of an arc at the boundary plus a vertical line pointing downwards to the spin a.

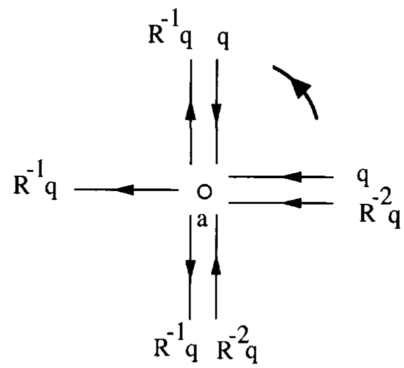

Fig. 11. Rotations of the line segment q.

Fig. 11. We can then make two further 90° rotations, as indicated, until the line is beneath the central site a, pointing downwards.

Again the line is anti-parallel to the vertical background lines and we can rotate (in this case push the line to the right) no further. Once more we use the equivalence of Fig. 8, replacing the rapidity now by $R^{-2}q$. One more 90° rotation returns the line to its original position and direction. However, q has been replaced by $R^{-2}q$ and the line segment has made a complete widdershins rotation round a. Allowing for these rotations, all the single-line segment configurations in Fig. 11 have the same probability.

Similar arguments apply to rotating p: in fact valid figures for this line segment can be obtained from Fig. 11 simply by replacing q by Rp.

We now use these rotation invariances to obtain three functional relations satisfied by $F_{pq}(a)$. From Fig. 11, all the configurations of the two special line segments shown in Figs. 12 and 13 have the same probability $F_{pq}(a)$. However, the first configuration in Fig. 12 can obviously be obtained from the third by merely replacing p, q by R^2p, R^2q, so

$$F_{R^2p,\, R^2q}(a) = F_{pq}(a) \tag{14}$$

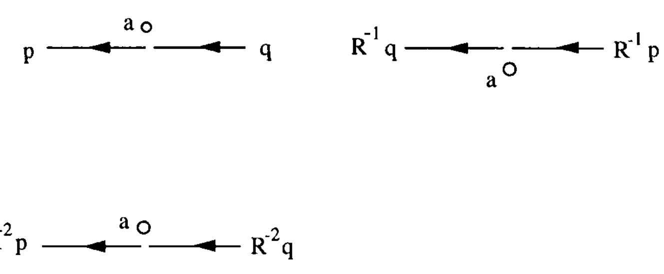

Fig. 12. Three configurations of the special line segments all with probability $F_{pq}(a)$.

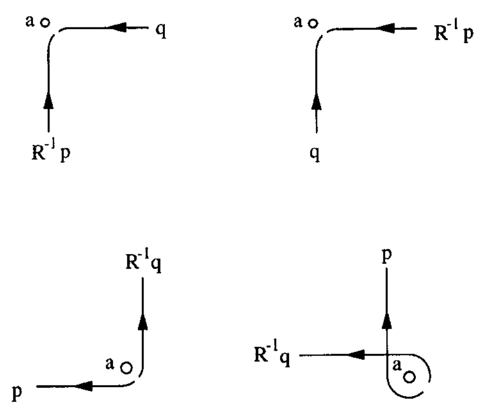

Fig. 13. More configurations of the special line segments with probability $F_{pq}(a)$.

Similarly, the first configuration in Fig. 13 can be obtained from the second by replacing p, q by Rq, $R^{-1}p$, so (replacing p by Rp)

$$F_{Rq,\,p}(a) = F_{Rp,\,q}(a) \tag{15}$$

The fourth configuration in Fig. 13 differs from the third in that the special line has wrapped a complete additional turn round the spin a (this turn being inside any background lines round a). This means that a is one-valent: it's only neighbour is the site designated as b in the figure. From Fig. 4, the edge between a and b has weight function $\bar{W}_{p,\,R^{-1}q}(a, b)$. If spin a and this edge is removed, the remaining graph is the same as the third configuration, but with p, q, a replaced by $R^{-1}q$, Rp, b. Replacing q by Rq, it follows that

$$F_{p,\,Rq}(a) = \xi_{pq} \sum_{b=0}^{N-1} S(b)\, \bar{W}_{pq}(a, b)\, F_{q,\,Rp}(b) \tag{16}$$

where ξ_{pq} is some normalization factor, independent of a.

In Fig. 14 we have drawn the first configuration of Fig. 13, including the background rapidity lines nearest to the spin a, and the nearby sites and edges of $\mathscr{G}$.

3.2. Z_N-Symmetric Models

The above remarks apply to any edge-interaction lattice model which is "solvable" in the sense that it satisfies the Yang–Baxter relations, more

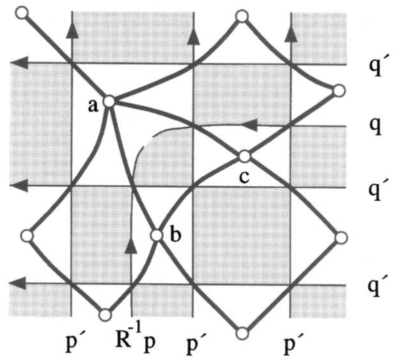

Fig. 14. The first configuration of Fig. 13, showing the background rapidity lines and the graph G.

precisely the star–triangle relations (4) and the inversion relations (5). For example, they apply to the Kashiwara–Miwa model.[27]

From now on we specialize to models which are Z_N-symmetric, i.e., $W_{pq}(a, b)$ and $\bar{W}_{pq}(a, b)$ depend on a and b only via their difference, modulo N:

$$W_{pq}(a, b) = W_{pq}(a-b), \qquad \bar{W}_{pq}(a, b) = \bar{W}_{pq}(a-b) \tag{17}$$

where $W_{pq}(n) = W_{pq}(n+N)$ and $\bar{W}_{pq}(n) = \bar{W}_{pq}(n+N)$, for all integers n. Also,

$$S(a) = 1$$

We continue to allow the possibility that the model is chiral, i.e., $W_{pq}(n) \neq W_{pq}(N-n)$ and $\bar{W}_{pq}(n) \neq \bar{W}_{pq}(N-n)$.

In this case the second of the star–triangle relations can be obtained from the first by negating all the spins (more strictly, by replacing each spin a by $N-1-a$), so there is only one such relation. From (9), $W_{Rp,\,Rq}(n) = W_{pq}(-n)$ and $\bar{W}_{Rp,\,Rq}(n) = \bar{W}_{pq}(-n)$, so replacing each rapidity p by Rp is equivalent to negating all spins (modulo N). It follows that

$$F_{Rp,\,Rq}(a) = F_{pq}(-a) \tag{18}$$

This relation implies (14), but not vice-versa, so for a Z_N-invariant system we can take the rotation symmetries to be equations (15)–(18).

From (10), $\sum_a F_{pq}(a) = 1$. Summing (16) over a, it follows that

$$\xi_{pq} = \sum_{b=0}^{N-1} \bar{W}_{pq}(b) \tag{19}$$

Fourier Transformed Equations. Let $\omega = \exp(2\pi i/N)$ and define, for all integers r,

$$\begin{aligned} \tilde{F}_{pq}(r) &= \sum_{a=0}^{N-1} \omega^{ra} F_{pq}(a) \\ \bar{V}_{pq}(r) &= \sum_{a=0}^{N-1} \omega^{ra} \bar{W}_{pq}(a) \end{aligned} \tag{20}$$

Then $\bar{F}_{pq}(0) = 1$ and the rotation symmetries can be written in the Fourier transformed form

$$\begin{aligned} \tilde{F}_{Rp,\, Rq}(r) &= \tilde{F}_{pq}(N-r) \\ \tilde{F}_{p,\, Rq}(r) &= \bar{V}_{pq}(r)\, \tilde{F}_{q,\, Rp}(r)/\bar{V}_{pq}(0) \\ \tilde{F}_{Rq,\, p}(r) &= \tilde{F}_{Rp,\, q}(r) \end{aligned} \tag{21}$$

Without loss of generality, in these equations we can take $r = 1,\dots, N-1$.

Also, from (13), the expectation value of ω^{ra} (for a spin a deep inside the homogeneous lattice) is

$$\mathcal{M}_r = \langle \omega^{ra} \rangle = \tilde{F}_{pp}(r) \tag{22}$$

and is independent of p.

It can be convenient to work not with $\tilde{F}_{pq}(1),\dots, \tilde{F}_{pq}(N-1)$, but with their ratios:

$$G_{pq}(r) = \tilde{F}_{pq}(r)/\tilde{F}_{pq}(r-1) \tag{23}$$

Then the equations become, for $r = 1,\dots, N$,

$$\begin{aligned} G_{Rp,\, Rq}(r) &= 1/G_{pq}(N-r+1) \\ G_{p,\, Rq}(r) &= \bar{V}_{pq}(r)\, G_{q,\, Rp}(r)/\bar{V}_{pq}(r-1) \\ G_{Rq,\, p}(r) &= G_{Rp,\, q}(r) \end{aligned} \tag{24}$$

We also have the normalization condition

$$\prod_{r=1}^{N} G_{pq}(r) = 1 \tag{25}$$

and

$$\mathscr{M}_r = G_{pp}(1) \cdots G_{pp}(r) \tag{26}$$

4. EQUATIONS FOR THE CHIRAL POTTS MODEL

The above remarks apply to any Z_N-invariant edge interaction model satisfying the star–triangle and inversion relations (4), (5). For instance they apply to the critical case of the original Potts model (Section 12.5 of ref. 10). Now we focus on the model that interests us here: the chiral Potts model. Let k, k' be two real constants (moduli) such that

$$k^2 + k'^2 = 1 \tag{27}$$

and let a_p, b_p, c_p, d_p be four complex numbers satisfying the homogeneous relations:

$$a_p^N + k'b_p^N = kd_p^N, \qquad k'a_p^N + b_p^N = kc_p^N \tag{28}$$

It follows that they also satisfy

$$ka_p^N + k'c_p^N = d_p^N, \qquad kb_p^N + k'd_p^N = c_p^N \tag{29}$$

We take p to be the point (a_p, b_p, c_p, d_p) on this homogeneous curve. Take q to be another such point (a_q, b_q, c_q, d_q). Then, from ref. 6,

$$W_{pq}(n) = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j}, \qquad \bar{W}_{pq}(n) = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega^j} \tag{30}$$

From this and (20), it follows that

$$\bar{V}_{pq}(r)/\bar{V}_{pq}(r-1) = (c_p b_q - a_p d_q \omega^r)/(b_p c_q - d_p a_q \omega^r)$$

so the second equation in (24) becomes

$$G_{p,Rq}(r) = \frac{c_p b_q - a_p d_q \omega^r}{b_p c_q - d_p a_q \omega^r} G_{q,Rp}(r) \tag{31}$$

The operator R is defined by

$$(a_{Rp}, b_{Rp}, c_{Rp}, d_{Rp}) = (b_p, \omega a_p, d_p, c_p) \tag{32}$$

We shall also use the operator M:

$$(a_{Mp}, b_{Mp}, c_{Mp}, d_{Mp}) = (\omega a_p, b_p, c_p, \omega d_p) \tag{33}$$

It satisfies the relation $RM = M^{-1}R$.

Replacing q by Mq in (30) merely multiplies $W_{pq}(n)$ by ω^{-n}, and $\bar{W}_{pq}(n)$ by ω^n. It follows that replacing p by Mp in Fig. 9 multiplies the partition function $Z(a)$ by ω^a. The same is true if we replace q in the figure by $M^{-1}q$. From (10), (20), (23) we therefore obtain the additional relations

$$\begin{aligned} F_{Mp,q}(a) &= F_{p,M^{-1}q}(a) = \omega^a F_{pq}(a)/\tilde{F}_{pq}(1) \\ \tilde{F}_{Mp,q}(r) &= \tilde{F}_{p,M^{-1}q}(r) = \tilde{F}_{pq}(r+1)/\tilde{F}_{pq}(1) \\ G_{Mp,q}(r) &= G_{p,M^{-1}q}(r) = G_{pq}(r+1) \end{aligned} \tag{34}$$

This completes our list of functional relations satisfied by $F_{pq}(a)$, $\tilde{F}_{pq}(r)$, and $G_{pq}(r)$. The remainder of this paper is concerned with attempts to solve them.

5. ISING CASE

When $N=2$, each spin has two possible states and the chiral Potts model reduces to the Ising model. In this case there is a straightforward and useful parametrization of the functional relations in terms of Jacobi's elliptic functions (Section 3 of ref. 11). For any p there exists a complex number u_p such that

$$a_p, b_p, c_p, d_p = -H(u_p), -H_1(u_p), \Theta_1(u_p), \Theta(u_p) \tag{35}$$

where $H(u)$, $H_1(u)$, $\Theta_1(u)$, $\Theta(u)$ are the elliptic theta functions of modulus k and argument u. The restrictions (28), (29) are then satisfied, for all u_p. Substituting these expressions into (30) and setting $u = u_q - u_p$, we obtain

$$W_{pq}(1) = k' \operatorname{scd}(K-u), \qquad \bar{W}_{pq}(1) = k' \operatorname{scd}(u) \tag{36}$$

where $\mathrm{scd}(u)=\mathrm{sn}(u/2)/[\mathrm{cn}(u/2)\,\mathrm{dn}(u/2)]$. If J, $\bar{J}$ are the usual (dimensionless) Ising model interaction coefficients, then $W_{pq}(1)=\exp(-2J)$, $\bar{W}_{pq}(1)=\exp(-2\bar{J})$ and

$$\sinh 2J=\frac{\mathrm{sn}\,u}{\mathrm{cn}\,u}, \qquad \sinh 2\bar{J}=\frac{\mathrm{cn}\,u}{k'\,\mathrm{sn}\,u} \tag{37}$$

and $\sinh 2J \sinh 2\bar{J}=1/k'$. Also, using (7),

$$u_{Rp}=u_p+K, \qquad u_{Mp}=u_p+2iK' \tag{38}$$

$$f_{pq}=\frac{H_1(0)\,\Theta_1(0)\,H_1[(K-u)/2]\,\Theta_1[(K-u)/2]}{H_1(K/2)\,\Theta_1(K/2)\,H_1(u/2)\,\Theta_1(u/2)}$$

$$\frac{c_p b_q+a_p d_q}{b_p c_q+d_p a_q}=\mathrm{sdc}(K-u) \tag{39}$$

where $\mathrm{sdc}(u)=\mathrm{sn}(u/2)\,\mathrm{dn}(u/2)/\mathrm{cn}(u/2)$.

Note that u_p and u_q enter $W_{pq}(1)$ and $\bar{W}_{pq}(1)$ only via their difference $u=u_q-u_p$ (this "difference property" holds for most planar models, but fails for the general chiral Potts model with $N>2$). The same must therefore be true for $F_{pq}(a)$, $\tilde{F}_{pq}(j)$ and $G_{pq}(j)$, so we can define a function $G(u)$ such that

$$G_{pq}(1)=1/G_{pq}(2)=G(u) \tag{40}$$

The functional relations (24), (25), (31), (34) become

$$G(K+u)=\mathrm{sdc}(K-u)\,G(K-u)$$

$$G(-K-u)=G(-K+u), \qquad G(u-2iK')=1/G(u) \tag{41}$$

and the spontaneous magnetization is

$$\mathscr{M}_1=G(0) \tag{42}$$

5.1. Zero-Temperature Limit

When k' is small and $0<\mathrm{Re}(u_q-u_p)<K$, then $W_{pq}(1)$ and $\bar{W}_{pq}(1)$ are also small:

$$W_{pq}(1)=e^{-\pi u/(2K')}, \qquad \bar{W}_{pq}(1)=e^{\pi(u-K)/(2K')} \tag{43}$$

and $K' \to \pi/2$. If $u_q - u_{p'}$, $u_p - u_{p'}$, $u_{q'} - u_{p'}$ in Fig. 9 all have real part between 0 and K, then the dominant contributions to $F_{pq}(a)$ come from all spins other than a being zero.

It follows that, for $-K < \operatorname{Re}(u_q - u_p) < K$

$$F_{pq}(0) = 1, \qquad F_{pq}(1) = W_{p'p}(1)\, W_{p'q'}(1)\, \bar{W}_{p'q}(1)\, \bar{W}p'q'(1)$$
$$= e^{-\pi(2K - u_q + u_p)/(2K')} \tag{44}$$

(Note that $u_{p'}$ and $u_{q'}$ cancel out of $F_{pq}(a)$, as they should.) Hence $\tilde{F}_{pq}(1) = 1 - 2e^{-\pi(2K - u_q + u_p)/(2K')}$ and, for $-K < \operatorname{Re} u < K$,

$$G(u) = 1 - 2e^{-\pi(2K - u)/(2K')} \tag{45}$$

Using similar arguments for the first and third of the special line configurations shown in Fig. 13, we find that to leading order $\log G(u)$ is given by the single formula

$$\log G(u) = -2e^{-\pi(2K-u)/(2K')} - 2e^{-\pi(4K+u)/(2K')} \tag{46}$$

over the larger range $-2K < \operatorname{Re} u < 2K$. (The last term comes from the first configuration, where we also have to include the contribution to $F_{pq}(1)$ from the state where spins b and c in Fig. 14 are both one.)

We can in principle develop a series expansion directly for $F_{pq}(1)/F_{pq}(0)$: each terms will be proportional to $e^{n\pi u/(2K')}$, where n is an odd (positive or negative) integer.

5.2. Nonzero Temperature: Solution for $G(u)$

The above remarks suggest that $\log G(u)$ may be analytic in the vertical strip $-K \leqslant \operatorname{Re} u \leqslant K$. We assume this.

The functional relations (also known as difference equations) (41) then (*and only then*) determine $G(u)$ uniquely. One way to solve them is to note that $\log G(u)$ is anti-periodic of period $2iK'$, so has a Fourier expansion of the form

$$\log G(u) = \sum_n{}' \, (g_n z^n + g_{-n} z^{-n}) \tag{47}$$

where

$$z = e^{\pi u/(2K')} \tag{48}$$

and the prime indicates that the sum (and all sums in this sub-section) is over positive odd integers n, i.e., $n = 1, 3, 5, \ldots$. The coefficients g_n, g_{-n} are to be determined.

From 8.146.20–22 of ref. 30, for $0 < \operatorname{Re} u < 2K$,

$$\log[\operatorname{sdc}(u)] = 2 \sum_n{}' \frac{x^n z^n - z^{-n}}{n(1+x^n)} \tag{49}$$

where

$$x = e^{-\pi K/K'} \tag{50}$$

Taking logarithms of (41) and equating Fourier coefficients, we can solve for g_n and g_{-n} to obtain

$$\log G(u) = -2 \sum_n{}' \frac{x^n z^n + x^{2n} z^{-n}}{n(1-x^n)(1+x^n)^2} \tag{51}$$

provided $-4K < \operatorname{Re} u < 2K$.

It follows that

$$\log G(0) = -2 \sum_n{}' \frac{x^n}{n(1-x^{2n})} = \sum_n{}' \log \frac{1-x^n}{1+x^n} \tag{52}$$

So from (42) and Eq. 8.197.4 of ref. 30

$$\mathscr{M}_1 = k^{1/4} = (1-k'^2)^{1/8} \tag{53}$$

which is the Onsager–Yang result[31, 32] for the spontaneous magentization of the Ising model.

Because of the "double pole" factor $(1+x^n)^2$ in (51), $G(u)$ is *not* periodic of period $2K$, or $4K$, or any integer multiple of K. However, the product $G(u)\, G(-u)$ is. In fact

$$G(u)\, G(-u) = k^{1/2} \operatorname{cn}(u/2)/\operatorname{dn}(u/2) \tag{54}$$

so $G(u)\, G(-u)$ is anti-periodic of period $4K$. This suggests that this product may be an algebraic function of the original parameters $a_p, \ldots, d_q$, and indeed we can verify that

$$G(u)\, G(-u) = \left\{ k \frac{b_p d_q + d_p b_q}{c_p d_q + d_p c_q} \; \frac{c_p a_q + a_p c_q}{b_p a_q + a_p b_q} \right\}^{1/2} \tag{55}$$

The above result $\mathscr{M}_1 = G(0) = k^{1/4}$ follows immediately.

6. A PRODUCT FUNCTION

Let us define one more function

$$L_{pq}(r) = G_{pq}(r)\, G_{Rq,\, Rp}(r) \tag{56}$$

From Fig. 6, replacing p, q by Rq, Rp is equivalent to taking the special rapidity lines to be above the spin a, rather than below, so $L_{pq}(r)$ is in some sense a product of these two situations. In fact, if one considers two independent chiral Potts models, one on the graph $\mathscr{G}$, and the other on the graph formed by putting spins on the shaded (rather than unshaded) faces of $\mathscr{L}$, one can then formulate a vertex model on $\mathscr{L}$ by assigning states to the edges of $\mathscr{L}$ which are the sum (or an appropiately ordered difference) of the spins on either side of the edge (much as the eight-vertex model can be formed from two Ising models—Section 10.3 of ref. 10—but without the four-spin interaction between them). If one does this, one obtains precisely $L_{pq}(r)$ as the ratio of the Fourier transforms of the edge correlation functions.

From (24), (25), (31), (34), (26) we obtain the following functional relations for $L_{pq}(r)$ and $\mathscr{M}_r$, for $r = 1,..., N$, with $L_{pq}(N+1) = L_{pq}(1)$:

$$\begin{aligned}
L_{Rp,\, Rq}(r) &= L_{qp}(r) = 1/L_{pq}(N-r+1)\\
L_{p,\, Rq}(r) &= \frac{c_p b_q - a_p d_q \omega^r}{b_p c_q - d_p a_q \omega^r}\, L_{q,\, Rp}(r)\\
\prod_{r=1}^{N} L_{pq}(r) &= 1\\
L_{Mp,\, q}(r) &= L_{p,\, M^{-1}q}(r) = L_{pq}(r+1)\\
\mathscr{M}_r^2 &= L_{pp}(1) \cdots L_{pp}(r)
\end{aligned} \tag{57}$$

From the first relation, $L_{q,\, Rp}(r) = L_{p,\, R^{-1}q}(r)$. Making this substitution in the RHS of the second relation, then iterating N times, we obtain

$$L_{p,\, R^{2N}q}(r) = (-1)^{N-1}\, L_{pq}(r) \tag{58}$$

so $L_{pq}(r)$ is periodic or anti-periodic under $q \to R^{2N}q$. The same is true for $p \to R^{2N}p$. Neither is true for $G_{pq}(r)$.

For $N=2$, $L_{pq}(1) = G(u)\, G(-u)$, so $L_{pq}(1)$ is given by (55). These observations suggest that $L_{pq}(r)$ may be an algebraic function of $a_p,..., d_q$ for all N. More strongly, it may be a meromorphic function on an extended surface, where its values on different surfaces differ by only a sign or a phase factor ω^n, as in the $N=2$ Ising case.

6.1. Possible Zeros and Poles

Let us postulate a function $\Theta_{pq}(i, j, m)$ on the $a_p, \ldots, d_q$ surface, which is analytic and has a zero, which is simple, only when

$$\begin{aligned} a_q &= \xi\omega^i a_p, \qquad & b_q &= \xi\omega^j b_p \\ c_q &= \xi\omega^m c_p, \qquad & d_q &= \xi d_p \end{aligned} \tag{59}$$

Here ξ is an uninteresting normalization factor. The function $\Theta_{pq}(i, j, m)$ may be multiple-valued, but its values should differ only by non-zero analytic factors. (The precise nature and even the existence of these functions is not really significant here. We are merely using them as a device to count the poles and zeros of $L_{pq}(r)$.) Then, to within non-zero analytic factors,

$$\begin{aligned} \Theta_{Rp, Rq}(i, j, m) &= \Theta_{pq}(j-m, i-m, -m) \\ \Theta_{qp}(i, j, m) &= \Theta_{pq}(-i, -j, -m) \\ \Theta_{R^2p, q}(i, j, m) &= \Theta_{pq}(i+1, j+1, m) \\ \Theta_{Mp, q}(i, j, m) &= \Theta_{p, M^{-1}q}(i, j, m) = \Theta_{pq}(i, j-1, m-1) \end{aligned}$$

Then, for instance, $\Theta_{pq}(i, j, m)$ is a factor of $c_p a_q - \omega^r a_p c_q$ when and only when $i - m = r$, so to within non-zero analytic factors

$$c_p a_q - \omega^r a_p c_q = \prod_{j=0}^{N-1} \prod_{m=0}^{N-1} \Theta_{pq}(m+r, j, m) \tag{60}$$

In this way one finds the $N=2$ result (55) can be written

$$L_{pq}(1) = \Theta_{pq}(1, 1, 0)/\Theta_{pq}(1, 0, 1) \tag{61}$$

For general N, we assume that $L_{pq}(r)$ only has poles or zeros at the zeros of the $\Theta_{pq}(i, j, m)$, so we try:

$$L_{pq}(r) = \prod_{i=0}^{N-1} \prod_{j=0}^{N-1} \prod_{m=0}^{N-1} \Theta_{pq}(i, j-r, m-r)^{\alpha(i, j, m)} \tag{62}$$

where each $\alpha(i, j, m)$ is an integer. The last of the relations (57) ensure that $\alpha(i, j, m)$ is independent of r.

Sustituting this ansatz into the remaining relations (57) we obtain

$$\begin{gathered}\alpha(m-j, m-i, m)=\alpha(i, j, m)=-\alpha(-i, 1-j, 1-m)\\ \alpha(i, j, m)=\alpha(i-1, j-1, m)+\delta_{i, m}-\delta_{j, 1}\\ \sum_{m} \alpha(i, j+m, m)=0\end{gathered} \tag{63}$$

for all integers i, j, m, mod N, the sum being over $m=0,\ldots, N-1$. This is a linear set of equations. Its general solution is any particular solution plus the general solution of the homogeneous relations, which implies that there exist functions (not necessarily integers) $\beta(i, j), \gamma(i, j)$ such that

$$\alpha(i, j, m)=\beta(i-m+1, j)+\gamma(i-j+1, m) \tag{64}$$

where

$$\begin{gathered}\beta(i, j)=-\beta(j, i)=\beta(1-j, 1-i)=\beta(i-1, j-1)+\delta_{i, 1}-\delta_{j, 1}\\ \gamma(i, j)=-\gamma(1-i, 1-j), \qquad \sum_{m} \beta(i-m, m)=0, \qquad \sum_{m} \gamma(i-m, m)=0\end{gathered} \tag{65}$$

for all integers i, j, mod N.

Let β, γ be the N by N matrices with elements $\beta(i, j), \gamma(i, j)$, for $i, j=0,\ldots, N-1$. Then for $N=2$, the general solution of (65) is

$$\beta=\begin{pmatrix}0 & -1/2\\ 1/2 & 0\end{pmatrix}, \qquad \gamma=\begin{pmatrix}v & u+1/2\\ -u-1/2 & -v\end{pmatrix} \tag{66}$$

(e.g., $\gamma(1,0)=-v-1/2$). here u and v are arbitrary integer parameters. We can fix one of them by noting that we expect $L_{pp}(r)$ to be finite and non-zero, so cannot contain a factor $\Theta_{pq}(0,0,0)$ and hence

$$\alpha(0,0,0)=0 \tag{67}$$

This implies $u=0$. The number of poles and zeros of $L_{pq}(r)$ is minimized by choosing $v=0$, giving

$$L_{pq}(r)=\Theta_{pq}(1, -r, 1-r)/\Theta_{pq}(1, 1-r, -r) \tag{68}$$

and indeed this is the known result (61).

For $N=3$, the general solution of (65) is

$$\beta=\begin{pmatrix} 0 & -s-1 & s \\ 1+s & 0 & -s \\ -s & s & 0 \end{pmatrix}, \qquad \gamma=\begin{pmatrix} w-v & u & -v \\ -u & v-w & v \\ w & -w & 0 \end{pmatrix} \tag{69}$$

There is a redundancy in this: when we substitute into (64) we find that s, u, v, w occur only via their differences, so without loss of generality we can choose $s=0$. The requirement (67) implies that $u=1$, so we are left with just v and w as arbitrary variables.

At this stage we have no further information, so all we can do is to look for the "simplest" solution for the 27 integers $\alpha(i, j, m)$. Of these, 5 are already constrained to be zero, 4 are ± 1, 4 are $\pm v$, 4 are $\pm w$, 2 are $\pm(v-1)$, 2 are $\pm(w-1)$ and 6 are $\pm(v-w)$. If we arbitrarily require that $L_{pq}(r)$ have only simple poles and zeros, then v, w can only take the values 0 or 1. This gives four possible cases. Writing $\Theta_{pq}(i, j, m)$ simply as (i, j, m), the two simplest are are:

(i) $v=w=0$: $L_{pq}(r)$ has four poles and four zeros:

$$L_{pq}(0)=\frac{(1,0,1)(1,2,1)(2,0,1)(2,0,2)}{(1,1,0)(1,1,2)(2,1,0)(2,2,0)} \tag{70}$$

(ii) $v=w=1$: it has six poles and six zeros

$$L_{pq}(0)=\frac{(0,0,2)(0,2,0)(1,0,0)(1,2,1)(2,0,1)(2,2,2)}{(0,1,2)(0,2,1)(1,1,0)(1,2,2)(2,1,1)(2,2,0)} \tag{71}$$

The other two cases, $(v, w)=(0,1)$ and $(1,0)$ each have eight poles and eight zeros and will not be considered further here.

Both the $N=3$ case (i) and the known $N=2$ solution (55) are contained in the general-N formula $L_{pq}(r)=L_{pq}^{(0)}(r)$, where

$$L_{pq}^{(0)}(r)=k^{(N-1)/N}\prod_{m=1}^{N-1}\left\{\frac{b_p d_q-\omega^{r-m-1}d_p b_q}{c_p d_q-\omega^{r-m-1}d_p c_q}\times\frac{c_p a_q-\omega^{r-m-1}a_p c_q}{b_p a_q-\omega^{r-m-1}a_p b_q}\right\}^{m/N} \tag{72}$$

This satisfies the functional relations (57) and despite appearances is in fact meromorphic of the form (62), with

$$L_{pq}^{(0)}(r)=\prod_{i=1}^{N-1}\prod_{j=i+1}^{N}\prod_{m=1}^{i}\Theta_{pq}(i, j-r, m-r)/\Theta_{pq}(i, m-r, j-r) \tag{73}$$

At first sight this seems (and seemed) very exciting: an explicit general-N solution which includes the Ising result. Unfortunately it's wrong. From (28),

$$c_p^N d_q^N - d_p^N c_q^N = b_p^N a_q^N - a_p^N b_q^N = k(b_p^N d_q^N - d_p^N b_q^N) = k(c_p^N a_q^N - a_p^N c_q^N) \tag{74}$$

Using these relations, we find that $L_{pp}^{(0)}(r) = k^{(N+1-2r)/N}$ for $r = 1,..., N$, which gives

$$\mathscr{M}_r = k^{r(N-r)/(2N)} = (1-k'^2)^{r(N-r)/(4N)} \tag{75}$$

This differs from the conjecture (1) by a power $N/2$, and this difference is manifest at order k'^2. Since the conjecture is certainly correct to this order, (72) must be *wrong* for $N > 2$: $L_{pq}(r) \neq L_{pq}^{(0)}(r)$.

Yet $L_{pq}^{(0)}(r)$ certainly satisfies the functional relations (57): the only explanation is that it cannot have the correct analyticity properties that are needed to complete the relations. This makes it clear that functional relations (or difference equations) do *not* by themselves uniquely define the functions: one must have extra insight into the analyticity properties in some fundamental domain. (Another example of this is provided by the three-dimensional Zamolodchikov model [23].)

Since $L_{pq}^{(0)}(r)$ is a solution (albeit not the right one) of the functional relations, if $L_{pq}(r)$ is of the form (62), then its ratio to $L_{pq}^{(0)}(r)$ is given by (62) with $\alpha(i, j, m)$ replaced by the homogeneous solution $\gamma(i-j+1, m)$. If we define

$$\Phi_{pq}(j, m) = \prod_{i=0}^{N-1} \Theta_{pq}(i, i-j, m) \tag{76}$$

(this is the function with a simple zero when $a_q/b_q = \omega^j a_p/b_p$ and $c_q/d_q = \omega^m c_p/d_p$), then it follows that

$$L_{pq}(r) = L_{pq}^{(0)}(r) \prod_{j=0}^{N-1} \prod_{m=0}^{N-1} \Phi_{pq}(j+r-1, m-r)^{\gamma(j,m)} \tag{77}$$

For the $N = 3$ case (ii) solution (71),

$$L_{pq}(r)/L_{pq}^{(0)}(r) = \frac{\Phi_{pq}(r+1, -r)\, \Phi_{pq}(r, 2-r)}{\Phi_{pq}(r+1, 1-r)\, \Phi_{pq}(r-1, 2-r)} \tag{78}$$

This still leaves open the question of whether case (ii) for $N=3$ correctly describes $L_{pq}(r)$, or whether we should be looking at the the other cases, or whether $L_{pq}(r)$ is indeed meromorphic, with poles and zeros as in (62). These problems are currently under investigation. In a subsequent paper the author intends to discuss the functional relations for the $N=3$ case in terms of the hyperelliptic function parametrization,[24, 25] and to present series expansions for $F_{pq}(a)$, $\tilde{F}_{pq}(j)$ that should shed some light on these problems.

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